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THE REPRESENTATION OF AN EMPIRICAL FUNCTION BY MEANS OF A POLYNOMIAL

AN A.E.R.E. REPORT

by

D.J. BEHRENS

MINISTRY OF SUPPLY,
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THE REPRESENTATION OF AN EMPIRICAL FUNCTION BY MEANS OF A POLYNOMIAL.

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D. J. Behrens

Abstract

If the values of a function are given at n equally-spaced intervals of the argument, a polynomial of order $n-1$ can be found which corresponds exactly to the given values. In general a polynomial of order $r(<n-1)$ cannot be made to do this, and $\mathcal{S}_r(f)$, the minimized sum of the squares of the deviations, will exceed 0.

It is shown in this paper how $\mathcal{S}_r(f)$, and the leading coefficient of the polynomial in question, can be evaluated without having to solve any simultaneous equations.

Contents

November 1950, with minor revisions July 1953.

H.D. 420

1. Introduction

It frequently happens in experimental work, that we can measure the value of some function f at several values of the argument x, and we wish to find a polynomial p(x) which is a sufficiently accurate measure of the given functional values.

We shall confine our attention to the case in which (1) equal errors are considered to be equally probable in each of the given functional values, so that the method of least squares may be used without weighting factors to give the "best" polynomial, and in which (2) the values of the function are given for equally spaced values of the argument x. We shall assume them to be given at $x = 1, 2, \ldots n$, and the corresponding functional values will be denoted by $f_1\ f_2 \ldots\ f_n$.

The two reservations made in the previous paragraph can usually be ensured by suitable experimental technique. We make them, because on their validity depends the simple method developed below.

It is clear that, if n functional values are given, we can construct a polynomial $p_{n-1}(x)$ of order n−1 which will pass exactly through each point. In general, a polynomial of order r, less than n−1, will not pass exactly through all the points. We can however define an r^{th} order polynomial $p_r(x)$ by adjusting its coefficients so as to minimize the sum of the squares

$$\sum_{j=1}^{n} \left\{p_r(j) - f_j\right\}^2 .$$

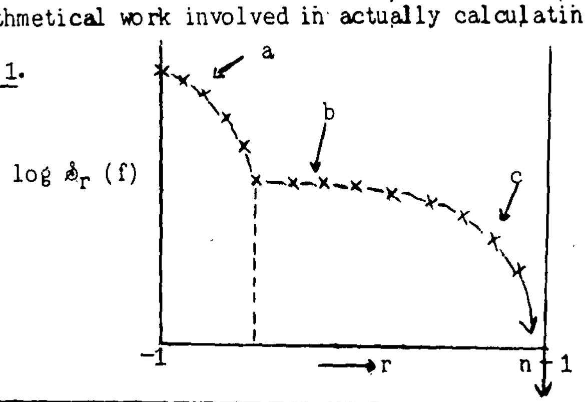
This minimized sum will be denoted by $\delta_r(f)$.

If now we examine the form of $\delta_r(f)$ as r is increased from -1* to n−1, we find that, if the functional values $f_1\ f_2 \ldots f_n$ really do lie close to a polynomial of comparatively low order, the general form of $\delta_r(f)$ is as sketched in Fig. 1.

In the region shown as (a), we have too few parameters to represent adequately even the general outline of the curve, while in (c) we have so many that phoney "accuracy" is obtained by fitting a curve even through the random errors of observation. The region (b) is that in which we have enough parameters to represent the general form of the curve, but not so many as to purport spurious accuracy. The value of r in which we are interested lies at the lower end of this region, and is indicated by the dotted line in Fig. 1.

We shall show how to find this value of r without the tedious and heavy arithmetical work involved in actually calculating p(x) for each r.

Figure 1.



* y = constant is a polynomial of order zero: it is convenient to define y = 0, which has one fewer degrees of freedom, as a polynomial of order −1.

2. Determination of $\delta_r(f)$.

Let Δ_j denote $f_{j+1} - f_j$, $\Delta_j^2 = \Delta_{j+1} - \Delta_j$, and in general $\Delta_j^r = \Delta_{j+1}^{r-1} - \Delta_j^{r-1}$. This is the usual finite-difference notation for forward differences.

Let $\mathfrak{m}_r$ denote the mean of the r^{th} order differences Δ^r, weighted in accordance with the formula

$$\mathfrak{m}_r = \sum_{j=1}^{n-r} \binom{n-j}{r}\binom{r+j-1}{r} \Delta_j^r \Big/ \binom{n+r}{2r+1}$$

where the terms in round brackets ( ) denote binomial coefficients. [It will be proved in appendix (proof 1) that $\sum_{j=1}^{n-r}\binom{n-j}{r}\binom{r+j-1}{r}$ is in fact equal to $\binom{n+r}{2r+1}$, so that $\mathfrak{m}_r$ defined as above is in fact a weighted mean of the Δ^r's.]

$\mathfrak{m}_0$ is defined in the same way, remembering that $\Delta_j^0 = f_j$, and in fact $\mathfrak{m}_0$ can be seen to be the arithmetic mean of the f's. In general, $\mathfrak{m}_r$ is the r^{th} difference of $p_r(x)$, the r^{th}-order polynomial giving the closest fit to the given f's (See Appendix, Proof 2).

Then $\delta_r(f)$ and $\delta_{r-1}(f)$ are connected by the relation

$$\delta_{r-1}(f) - \delta_r(f) = \binom{n+r}{2r+1} \mathfrak{m}_r^2 \Big/ \binom{2r}{r}$$

See Appendix, proofs 3 and 4.

Since we know that $\delta_{n-1}(f) = 0$, and that $\delta_{-1}(f) = \sum_{j=1}^{n} (f_j^2)$, we can calculate the δ_r's beginning from either end. Alternatively the fact that both end-values are known serves as a check on the arithmetic if we calculate $\delta_{r-1}(f) - \delta_r(f)$ for every r from 0 to n-1.

3. Probable error in $\mathfrak{m}_r$.

It remains to calculate the probable error of the $\mathfrak{m}$'s, in terms of that of the given functional values.

Expressing $\mathfrak{m}_r$ in terms of the f's themselves, instead of their r^{th} differences, we get

$$\mathfrak{m}_r = \sum_{j=1}^{n} f_j \sum_{m=0}^{r} (-1)^m \binom{r}{m}\binom{m+j-1}{r}\binom{n-j-m+r}{r} \Big/ \binom{n+r}{2r+1}$$

And thus the mean square error in $\mathfrak{m}_r$ is given in terms of that in the f's by

$$\delta^2 \mathfrak{m}_r / \delta^2 f = \frac{\sum_{j=1}^{n}\left\{\sum_{m=0}^{r} (-1)^m \binom{r}{m}\binom{m+j-1}{r}\binom{n-j-m+r}{r}\right\}^2}{\binom{n+r}{2r+1}^2}$$

This reduces to

$$\delta^2 \mathfrak{m}_r / \delta^2 f = \binom{2r}{r} \Big/ \binom{n+r}{2r+1} \qquad \text{[see Appendix, proof 5]}$$

But this ratio has already been shown to be equal to $\mathfrak{m}_r^2/(\mathcal{S}_{r-1}(f) - \mathcal{S}_r(f))$, so we have the result that

$$\frac{\delta^2 \mathfrak{m}_r}{\mathfrak{m}^2 r} = \frac{\delta^2 f}{\mathcal{S}_{r-1}(f) - \mathcal{S}_r(f)}$$

Often, we do not know the mean square error $\delta^2 f$. We may however assume that, if r has been sensibly chosen, it is of the order of $\mathcal{S}_r(f)/(n-r-1)$ – we divide by $(n-r-1)$ as this is the excess number of degrees of freedom over those required to fix the parameters of $p_r(x)$.

Thus we may say $\delta^2\mathfrak{m}_r/\mathfrak{m}_r^2 = \mathcal{S}_r(f)/(n-r-1)(\mathcal{S}_{r-1}(f) - \mathcal{S}_r(f))$, and the condition that $\mathfrak{m}_r$ should be significantly different from zero is $\mathcal{S}(f) < (n-r-1)(\mathcal{S}_{r-1}(f) - \mathcal{S}_r(f))$,

$$\text{or} \quad \mathcal{S}_r(f) \Big/ (n-r-1) < \mathcal{S}_{r-1}(f) \Big/ (n-r)$$

It is therefore worth while, in tabulating $\mathcal{S}_r(f)$, also to note the value of the quotient $\mathcal{S}_r(f)/(n-r-1)$, since (a) when $\delta^2 f$ is known, it is this quantity which must be kept down to the order of $\delta^2 f$, and (b) in any case if the quotient $\mathcal{S}_r(f)/(n-r-1)$ exceeds the corresponding value at $(r-1)$, this shows that no significant additional accuracy arises from taking an r^{th} order polynomial instead of p_{r-1}. This argument is used in Example 3 below.

Reference to other work

M. G. Smith "The Lorentz Method of Analysis of Experimental Data using Orthogonal Polynomials" (A.R.E. Report No. 15/52). [This paper presents a method for calculating the best-fitting r^{th} order polynomial, having first obtained that of order r-1. It differs from the treatment of the present paper in computing first the polynomials themselves, and only afterwards the values of $\mathcal{S}(f)$, whereas we have here treated the residual errors as being of prime interest – i.e. we determine the required order of polynomial without having first computed it. Either treatment can be more useful than the other, according to circumstances.]

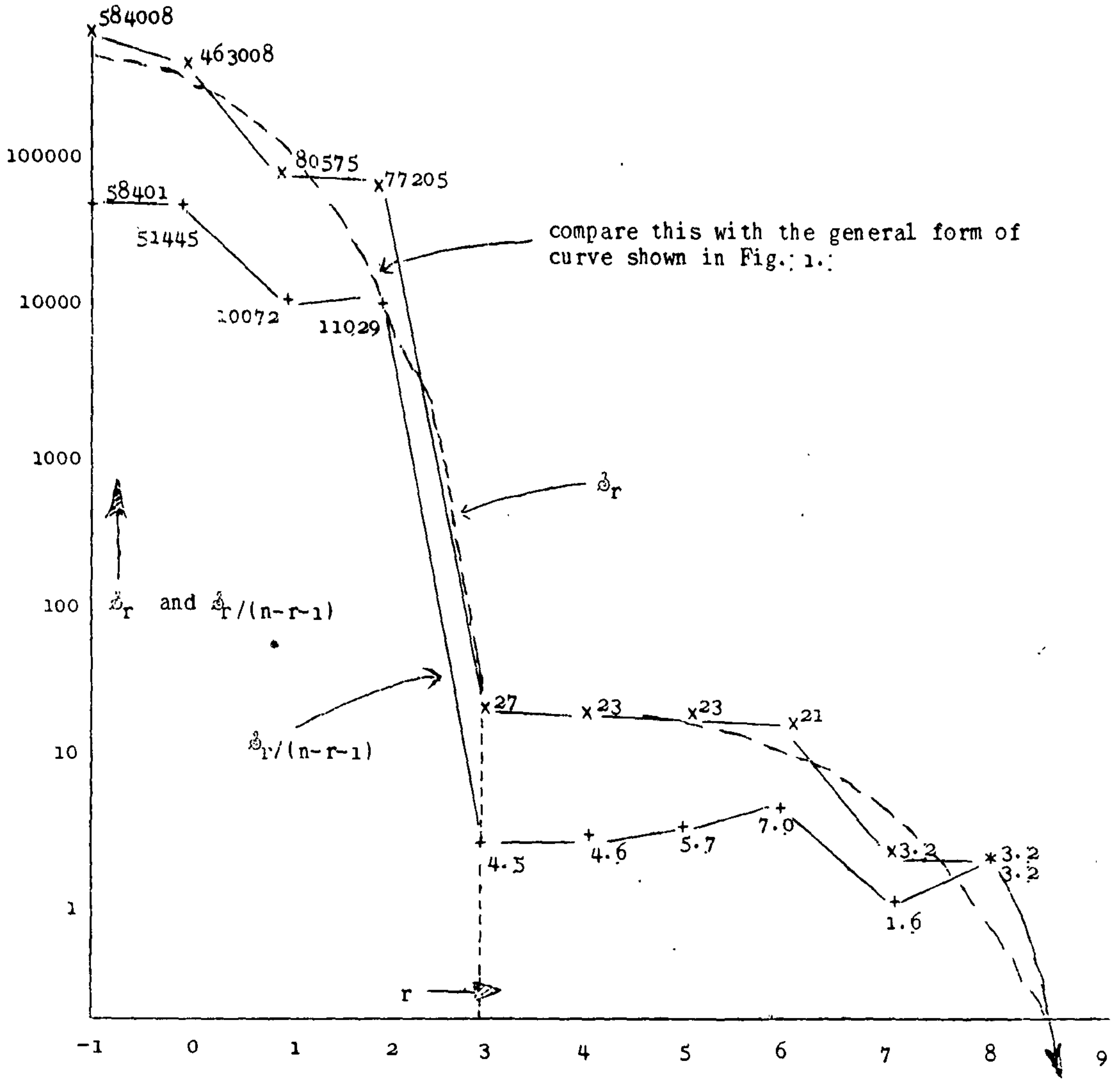
EXAMPLE I. Functional values lying close to a polynomial of low order (cubic).

r	1		2		3		4		5		6		7		8		9		10	SUMS
fr	261		378		392		320		198		63		−69		−162		−181		−100	
w. f.	1		1		1		1		1		1		1		1		1		1	10
	261		378		392		320		198		63		−69		−162		−181		−100	1100
Δ		117		14		−72		−122		−135		−132		−93		−19		81		
w. f.		9		16		21		24		25		24		21		16		9		165
		1053		224		−1512		−2928		−3375		−3168		−1953		−304		729		11234
Δ^2			−103		−86		−50		−13		3		39		74		100			
w. f.			36		84		126		150		150		126		84		36			792
			−3708		−7224		−6300		−1950		450		4914		6216		3600			4002
Δ^3				17		36		37		16		36		35		26				
w. f.				84		224		350		400		350		224		84				1716
				1428		8064		12950		6400		12600		7840		2184				51466
Δ^4					19		1		−21		20		−1		−9					
w. f.					126		350		525		525		350		126					2002
					2394		350		−11025		10500		−350		−1134					735
Δ^5						−18		−22		41		−21		−8						
w. f.						126		336		441		336		126						1365
						−2268		−7392		18081		−7056		−1008						357
Δ^6							−4		63		−62		13							
w. f.							84		196		196		84							560
							−336		12348		−12152		1092							952
Δ^7								67		−125		75								
w. f.								36		64		36								136
								2412		−8000		2700								−2888
Δ^8									−192		200									
									9		9									18
									−1728		1800									+72
Δ^9										392										
w. f.										1										1
										392										392

$\mathcal{S}_{-1} = \Sigma(f^2) = 584008$	$\frac{1}{10}\mathcal{S}_{-1} \approx 58401$
$\therefore\ \mathcal{S}_0 = 584008 - (1100)^2(10\times1) = 584008-121000 = 463008$	$\frac{1}{9}\mathcal{S}_0 \approx 51445$
$\mathcal{S}_1 = 463008-(11234)^2/(165\times2) = 463008-382432\frac{98}{165} = 80575\frac{67}{165}$	$\frac{1}{8}\mathcal{S}_1 \approx 10072$
$\mathcal{S}_2 = 80575\frac{67}{165}-(4002)^2/(792\times6) = 80575\frac{67}{165}-3370\frac{49}{132} = 77205\frac{23}{660}$	$\frac{1}{7}\mathcal{S}_2 \approx 11029$
$\mathcal{S}_3 = 77205\frac{23}{660}-(51466)^2/(1716\times20) = 77205\frac{23}{660}-77178\frac{48}{8580} = 27\frac{25}{858}$	$\frac{1}{6}\mathcal{S}_3 \approx 4.5$
$\mathcal{S}_4 = 27\frac{25}{858}-(735)^2/(2002\times70) = 27\frac{25}{858}-3\frac{409}{572} = 23\frac{23}{132}$	$\frac{1}{5}\mathcal{S}_4 \approx 4.6$
$\mathcal{S}_5 = 23\frac{23}{132}-(357)^2/(1365\times252) = 23\frac{23}{132}-\frac{289}{780} = 22\frac{1724}{2145}$	$\frac{1}{4}\mathcal{S}_5 \approx 5.7$
$\mathcal{S}_6 = 22\frac{1724}{2145}-(952)^2/(560\times924) = 22\frac{1724}{2145}-\frac{289}{780} = 21\frac{112}{2145}$	$\frac{1}{3}\mathcal{S}_6 \approx 7.0$
$\mathcal{S}_7 = 21\frac{112}{2145}-(2888)^2/(136\times3432) = 21\frac{112}{2145}-17\frac{101440}{116688} = 3\frac{171}{935}$	$\frac{1}{2}\mathcal{S}_7 \approx 1.6$
$\mathcal{S}_8 = 3\frac{171}{935}-(72)^2/(18\times12870) = 3\frac{171}{935}-\frac{16}{715} = 3\frac{1951}{12155}$	$\mathcal{S}_8 \approx 3.2$

$$\mathcal{S}_9 = 3\frac{1951}{12155}-(392)^2/(1\times48620) = 3\frac{1951}{12155}-3\frac{1951}{12155} = 0$$

EXAMPLE I (continued)



EXAMPLE II.

Functional Values Not Lying Close to a Polynomial of Low Order.

Independent variable:-	0		1		2		3		4		5		6	SUMS	
functional values:-	1		2		4		8		16		32		64		
weighting factors	1		1		1		1		1		1		1	7	$\mathfrak{m}_0 = \frac{127}{7}$
	1		2		4		8		16		32		64	127	
first differences		1		2		4		8		16		32			
weighting factors		6		10		12		12		10		6		56	$\mathfrak{m}_1 = \frac{261}{28}$
		6		20		48		96		160		192		522	
second differences			1		2		4		8		16				
weighting factors			15		30		36		30		15			126	$\mathfrak{m}_2 = \frac{233}{42}$
			15		60		144		240		240			699	
third differences				1		2		4		8					
weighting factors				20		40		40		20				120	$\mathfrak{m}_3 = \frac{7}{2}$
				20		80		160		160				420	
fourth differences					1		2		4						
weighting factors					15		25		15					55	$\mathfrak{m}_4 = \frac{25}{11}$
					15		50		60					125	
fifth differences						1		2							
weighting factors						6		6						12	$\mathfrak{m}_5 = \frac{3}{2}$
						6		12						18	
sixth differences							1								
weighting factors							1							1	$\mathfrak{m}_6 = 1$
							1							1	

$\delta_6 = 0$

$\delta_5 - \delta_6 = \frac{1}{924}\,\mathfrak{m}_6^2 = \frac{1}{924}$

$\delta_4 - \delta_5 = \frac{12}{252}\,\mathfrak{m}_5^2 = \frac{12}{252}\,\frac{9}{4} = \frac{3}{28}\ \therefore\quad \delta_4 = \frac{1}{924} + \frac{3}{28} = \frac{100}{924} = \frac{25}{231}$

$\delta_3 - \delta_4 = \frac{55}{70} \times \mathfrak{m}_4^2 = \frac{55}{70} \times \frac{625}{121} - \frac{625}{154} \qquad \delta_3 = \frac{25}{231} + \frac{625}{154} = \frac{1925}{462} = \frac{25}{6}$

$\delta_2 - \delta_3 = \frac{120}{20} \times \left(\frac{7}{2}\right)^2 = \frac{147}{2} \qquad \delta_2 = \frac{25}{6} + \frac{147}{2} = \frac{466}{6} = \frac{233}{3}$

$\delta_1 - \delta_2 = \frac{126}{6}\left(\frac{233}{42}\right)^2 = (233)^2/84 \qquad \delta_1 = 233\{\frac{1}{3} + \frac{233}{84}\} = \frac{20271}{28}$

$\delta_0 - \delta_1 = \frac{56}{2}\left(\frac{261}{28}\right)^2 = (261)^2/28 \qquad \delta_0 = (20271+68121)/28 = 22098/7$

$\delta_{-1} - \delta_0 = \frac{7}{1}\left(\frac{127}{7}\right)^2 = (127)^2/7 \qquad \delta_{-1} = (16129+22098)/7 = 5461$

$\delta_6 = 0 \approx$

$\delta_5 = \frac{1}{924} \approx 1.1 \times 10^{-3}$

$\delta_4 = \frac{25}{231} \approx 1.1 \times 10^{-1}$

$\delta_3 = \frac{25}{6} \approx 4.2$

$\delta_2 = \frac{233}{3} \approx 7.8 \times 10$

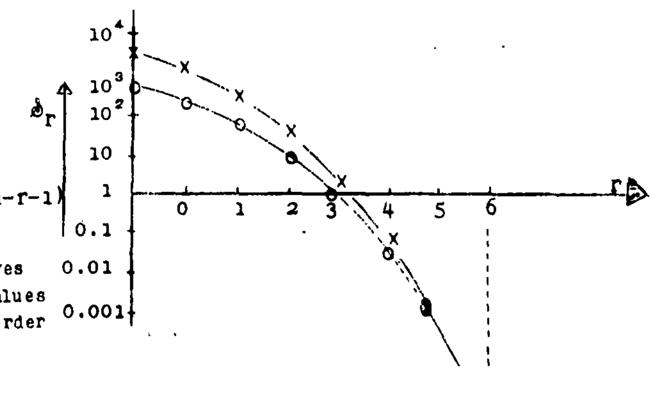
$\delta_1 = \frac{20271}{28} \approx 7.2 \times 10^2$

$\delta_0 = \frac{22098}{7} \approx 3.2 \times 10^3$

$\delta_{-1} = 5461 \approx 5.4 \times 10^3$

Check:- Sum of f^2 = 1 + 4 + 16 + 64 + 256 + 1024 + 4096 = 5461

and $\delta_{r/(n-r-1)}$



And the shape of the curves suggest that the functional values certainly do not fit any low-order polynomial.

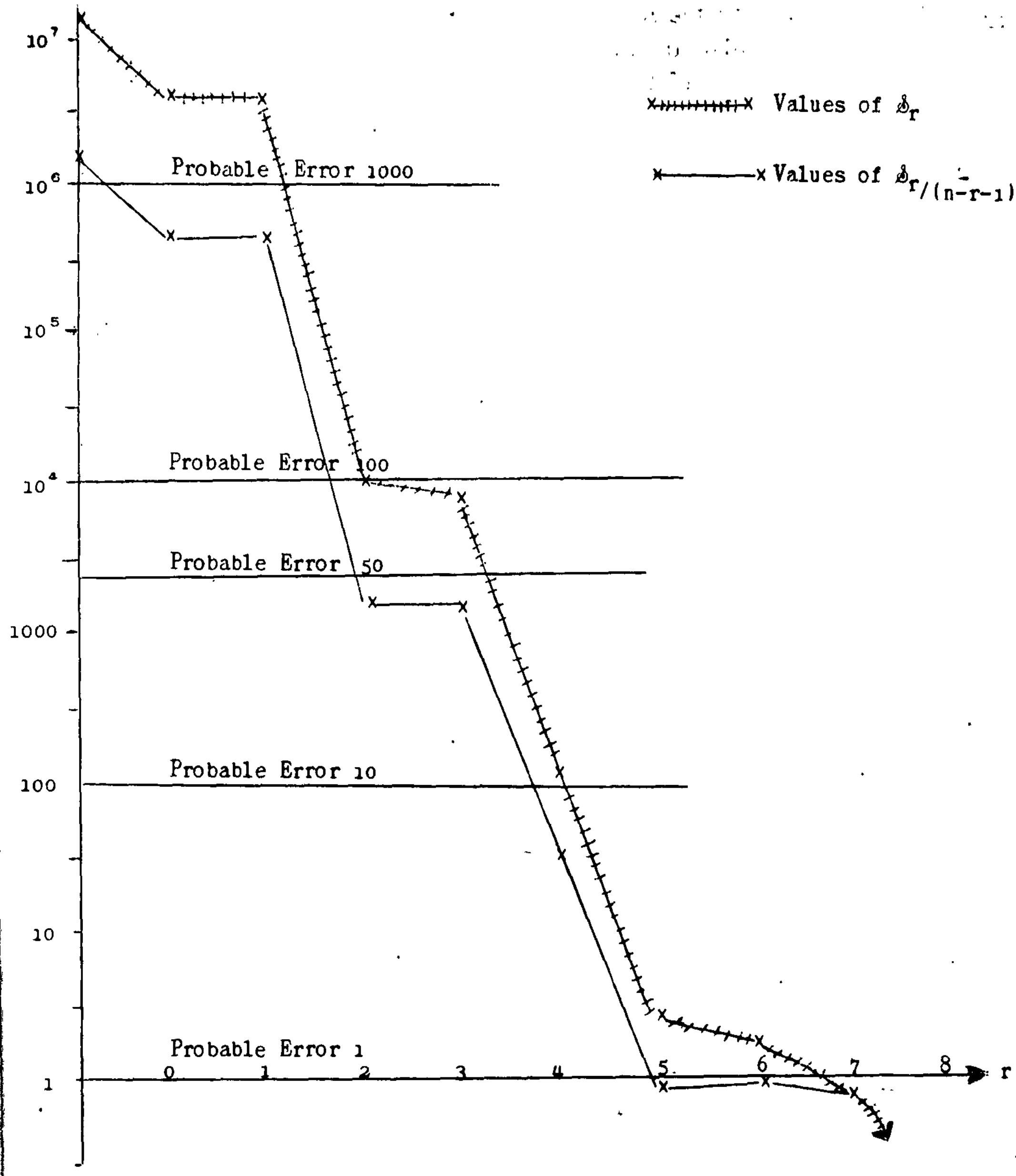
EXAMPLE III What is the lowest order of polynomial consistent with the following observations,

(a) if the probable error of each observation is 50
(b) " " " " " " " " 1

x	1	2	3	4	5	6	7	8	9
f(x)	27	676	1211	1576	1730	1640	1309	775	116

																		Σ(Δ wf)
f	27		676		1211		1576		1730		1640		1309		775		116	
wf	1		1		1		1		1		1		1		1		1	9060
Δ		649		535		365		154		−90		−331		−534		−659		
		8		14		18		20		20		18		14		8		1186
Δ^2			−114		−170		−211		−244		−241		−203		−125			
wf			28		63		90		100		90		63		28			− 95271
Δ^3				−56		−41		−33		3		38		78				
wf				56		140		200		200		140		56				− 5188
Δ^4					15		8		36		35		40					
wf					70		175		225		175		70					19475
Δ^5						−7		28		−1		5						
wf						56		126		126		56						3290
Δ^6							35		−29		6							
wf							28		49		28							− 273
Δ^7								−64		35								
wf								8		8								− 232
Δ^8									−99									
wf									1									− 99

$s_8 = 0$	$s_8 = 0$		
$s_7 = (99)^2/1 \times 12870 = 0.76$	$s_7 = 0.76$	$s_7 = 0.76$	$= (.87)^2$
$s_6 - s_7 = (232)^2/16 \times 3432 = 0.98$	$s_6 = 1.74$	$\frac{1}{2} s_6 = 0.87$	$= (.93)^2$
$s_5 - s_6 = (273)^2/105 \times 924 = 0.77$	$s_5 = 2.51$	$\frac{1}{3} s_5 = 0.84$	$= (.92)^2$
$s_4 - s_5 = (3290)^2/364 \times 252 = 118$	$s_4 = 121$	$\frac{1}{4} s_4 = 30$	$= (5.5)^2$
$s_3 - s_4 = (19475)^2/715 \times 70 = 7578$	$s_3 = 77 \times 10^2$	$\frac{1}{5} s_3 = 15.4 \times 10^2$	$= 39^2$
$s_2 - s_3 = (5188)^2/792 \times 20 = 1701$	$s_2 = 94 \times 10^2$	$\frac{1}{6} s_2 = 15.3 \times 10^2$	$= 39^2$
$s_1 - s_2 = (95271)^2/462 \times 6 = 32.7 \times 10^5$	$s_1 = 32.8 \times 10^5$	$\frac{1}{7} s_1 = 4.7 \times 10^5$	$= (690)^2$
$s_0 - s_1 = (1186)^2/120 \times 2 = 5861$	$s_0 = 32.9 \times 10^5$	$\frac{1}{8} s_0 = 4.1 \times 10^5$	$= (640)^2$
$s_{-1} - s_0 = (9060)^2/9 \times 1 = 91.2 \times 10^5$	$s_{-1} = 124.1 \times 10^5$	$\frac{1}{9} s_{-1} = 13.8 \times 10^5$	$= (1200)^2$



$\sqrt{\delta_r/(n-r-1)}$ should be of the same order of magnitude as the probable mean error of observation [for detailed treatment see any textbook of statistics, under the heading "χ^2 distribution of goodness of fit"].

In the example given, it will be seen that a parabola ($r = 2$) is consistent with a probable error of 50, while if the probable error is only unity, a 5th order polynomial is required to give an adequate fit.

With a probable error of 50 in each f, however, the lower values of $\sqrt{\delta_r/(n-r-1)}$ for r in excess of 3 would appear highly suspicious, and would lead very definitely to suppose that the errors are largely of a systematic rather than a random nature.

APPENDIX

Proof 1.

That $$\sum_{j=1}^{n-r}\binom{n-j}{r}\binom{r+j-1}{r} = \binom{n+r}{2r+1} \qquad [n > r].$$

Let $\delta(n,r)$ denote $$\sum_{j=1}^{n-r}\binom{n-j}{r}\binom{r+j-1}{r} - \binom{n+r}{2r+1}.$$

Writing the binomial coefficients as the sums of coefficients of lower order, we may say

$$\delta(n,r) = \sum_{j=1}^{n-r}\binom{n-j-1}{r-1}\left(1+\frac{n-j-r}{r}\right)\binom{r+j-2}{r-1}\left(1+\frac{j-1}{r}\right) - \binom{n+r-1}{2r}\left(1+\frac{n-r-1}{2r+1}\right)$$

$$= \sum_{j=1}^{n-r}\binom{n-j-1}{r-1}\binom{r+j-2}{r-1}\left(1+\frac{n-r-1}{r}+\frac{(n-j-r)(j-1)}{r^2}\right) - \binom{n+r-2}{2r-1}\left(1+\frac{n-r-1}{2r+1}\right)\left(1+\frac{n-r-1}{2r}\right)$$

The term $\frac{(n-j-r)(j-1)}{r^2}$ vanishes both at $j=1$ and at $j=n-r$; we may therefore shorten its range of summation by one at each end.

so that $$\delta(n,r) = \sum_{j=1}^{n-r}\binom{n-j-1}{r-1}\binom{r+j-2}{r-1}\left(1+\frac{n-r-1}{r}\right) + \sum_{j=2}^{n-r-1}\binom{n-j-1}{r}\binom{r+j-2}{r} - \binom{n+r-2}{2r-1}\left(1+\frac{n-r-1}{r}\right) - \binom{n+r-2}{2r+1}$$

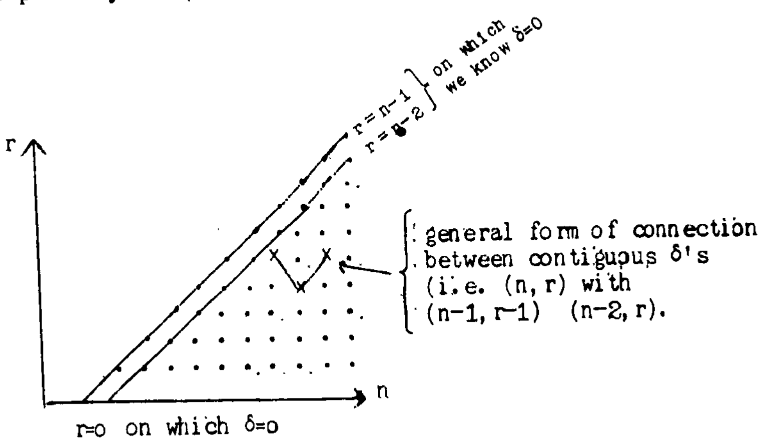
$$= \frac{n-1}{r}\,\delta(n-1,r-1) + \delta(n-2,r)$$

Now $$\delta(n,0) = \sum_{j=1}^{n}\binom{n-j}{0}\binom{j-1}{0} - \binom{n}{1} = n - n = 0$$

Again, at $r = n-1$, j can only be 1, and $\delta(n,n-1) = \binom{n-1}{n-1}\binom{n-1}{n-1} - \binom{2n-1}{2n-1} = 1 - 1 = 0$

At $r = n-2$ j can be 1 or 2, and $\delta(n,n-2) = \binom{n-1}{n-2}\binom{n-2}{n-2} + \binom{n-2}{n-2}\binom{n-1}{n-2} - \binom{2n-2}{2n-2} = 2(n-1) - (2n-2) = 0$

This suffices to prove by induction that $\delta(n,r) = 0$.



Proof 2.

That $\sum_{j=1}^{n-r} \binom{n-j}{r}\binom{r+j-1}{r} \Big/ \Delta_j^r \binom{n+r}{2r+1}$ is the r^{th} difference of the r^{th} order polynomial giving the closest fit to the points $f_1\ f_2 \ldots\ldots f_n$

Let the f's be chosen so that all but one of their r^{th} order differences vanish, and the other is unity. Let $\Delta_j^r = 1$, $\Delta_k^r = 0$ for $k \neq j$. Such a set of f's is given by $f_0 = f_1 \ldots = f_{j+r-1} = 0$ and $f_k = \binom{k-j-1}{r-1}$ for $k \geq j + r$.

Let $p_r(x) = a_r x^r + a_{r-1} x^{r-1} \ldots\ldots + a_1 x + a_0$.

We are to choose the a's so as to minimize

$$\sum_{x=1}^{j+r-1} (a_r x^r + a_{r-1}x^{r-1} + \ldots + a_1 x + a_0)^2 + \sum_{x=j+r}^{n} \left\{a_r x^r + a_{r-1} x^{r-1} + \ldots + a_1 x + a_0 - \binom{x-j-1}{r-1}\right\}^2$$

If we use a notation in which $\binom{g}{h} = 0$ for all $g < h$, we may simplify this, and say that we have to minimize

$$\sum_{x=1}^{n} \left\{ a_r x^r + a_{r-1} x^{r-1} + \ldots\ldots + a_1 x + a_0 - \binom{x-j-1}{r-1} \right\}^2$$

We obtain a set of simultaneous equations, from which $a_{r-1}\ a_{r-2} \ldots\ldots a_1\ a_0$ can be eliminated, leaving

$$a_r \begin{vmatrix} S_{2r} & S_{2r-1} & S_{2r-2} & \cdots & S_r \\ S_{2r-1} & S_{2r-2} & S_{2r-3} & \cdots & S_{r-1} \\ S_{2r-2} & S_{2r-3} & S_{2r-4} & \cdots & S_{r-2} \\ \hline S_r & S_{r-1} & S_{r-2} & \cdots & S_0 \end{vmatrix} = \begin{vmatrix} \sum_1^n x^r \binom{x-j-1}{r-1} & S_{2r-1} & S_{2r-2} & \cdots & S_r \\ \sum_1^n x^{r-1} \binom{x-j-1}{r-1} & S_{2r-2} & S_{2r-3} & \cdots & S_{r-1} \\ \sum_1^n x^{r-2} \binom{x-j-1}{r-1} & S_{2r-3} & S_{2r-4} & \cdots & S_{r-2} \\ \hline \sum_1^n \binom{x-j-1}{r-1} & S_{r-1} & S_{r-2} & \cdots & S_0 \end{vmatrix}$$

where S_k denotes $(1^k + 2^k + 3^k + \ldots\ldots + n^k)$.

We have to calculate the values of these determinants.

It may be shown that

$$\begin{vmatrix} \frac{1}{2r+1} & \frac{1}{2r} & \frac{1}{2r-1} & \cdots\cdots & \frac{1}{r+1} \\ \frac{1}{2r} & \frac{1}{2r-1} & \frac{1}{2r-2} & \cdots\cdots & \frac{1}{r} \\ \frac{1}{2r-1} & \frac{1}{2r-2} & \frac{1}{2r-3} & \cdots\cdots & \frac{1}{r-1} \\ - & - & - & - & - \\ \frac{1}{r+1} & \frac{1}{r} & \frac{1}{r-1} & \cdots\cdots & \frac{1}{1} \end{vmatrix} = [\mathfrak{Z}(r)]^4 / \mathfrak{Z}(2r+1)$$

where $\mathfrak{Z}(r)$ denotes the factorial function $1!\ 2!\ 2!\ 4!\ \ldots\ldots\ r!$

From this it follows that

$$\begin{vmatrix} S_{2r} & S_{2r-1} & \cdots & S_r \\ S_{2r-1} & S_{2r-2} & \cdots & S_{r-1} \\ - & - & - & - \\ S_r & S_{r-1} & \cdots & S_0 \end{vmatrix} = \mathfrak{Z}(n+r)\ \mathfrak{Z}(n-r-2)[\mathfrak{Z}(r)]^4/[\mathfrak{Z}(n-1)]^2\ \mathfrak{Z}(2r+1)$$

It can also be shown that

$$\begin{vmatrix} \sum_1^n x^r \binom{x-j-1}{r-1} & S_{2r-1} & \cdots & S_r \\ \sum_1^n x^{r-1} \binom{x-j-1}{r-1} & S_{2r-2} & \cdots & S_{r-1} \\ - & - & - & - \\ \sum_1^n \binom{x-j-1}{r-1} & S_{r-1} & \cdots & S_0 \end{vmatrix} = \left\{ r!\,\binom{n-j}{r}\binom{r+j-1}{r}\Big/\binom{2r}{r} \right\} \begin{vmatrix} S_{2r-2} & S_{2r-3} & \cdots & S_{r-1} \\ S_{2r-3} & S_{2r-4} & \cdots & S_{r-2} \\ - & - & - & - \\ S_{r-1} & S_{r-2} & \cdots & S_0 \end{vmatrix}$$

$$= \frac{\binom{n-j}{r}\binom{r+j-1}{r}\ \mathfrak{Z}(n+r-1)\ \mathfrak{Z}(n-r-1)\ \mathfrak{Z}(r-1)\ [\mathfrak{Z}(r)]^3}{[\mathfrak{Z}(n-1)]^2 \quad \mathfrak{Z}(2r)}$$

So that $a_r = \binom{n-j}{r} \binom{r+j-1}{r} \dfrac{(n-r-1)!\ (2r+1)!}{(n+r)!\ \ r!}$

And hence the m^{th} order difference of $p_r(x)$, which is $r!\,a_r$, is given by $\binom{n-j}{r} \binom{r+j-1}{r} \Big/ \binom{n+r}{2r+1}$. This is in fact the coefficient of Δ_j^r in the expression for $\mathfrak{m}_r$. As this applies for any j, and the differences and $\mathfrak{m}$'s are all linear in the f's, it follows that $\mathfrak{m}_r$ is always the r^{th} difference of the best $p_r(x)$, whatever be the values of the f's.

Proof 3. That $\dfrac{\delta_{r-1}(f) - \delta_r(f)}{\mathfrak{m}^2_r}$ is a function only of n and r.

Let f denote a set of functional values $f_1\ f_2\ f_3 \dots f_n$ given at n equally-spaced intervals of the argument.

Let $p_r(x)$ denote the best r^{th} order polynomial through the functional values – i.e. the parameters are so chosen that $\sum_{j=1}^{n}$ is minimized – and let $\delta_r(f)$ be used to denote this minimized sum, and $\mathfrak{m}_r(f)$ the r^{th} difference of the polynomial $p_r(x)$.

Now let f^* denote that set of functional falues $f^*_1\ f^*_2 \dots f^*_n$ such that the first (r+1) of them, viz f^*_1 to f^*_{r+1}, are the same as those in the f set ($f^*_1 = f_1$, $f^*_2 = f_2$, $\dots f^*_{r+1}$), and the remaining n–r–1 values are so chosen that they lie on the r^{th} order polynomial through $f_1, f_2 \dots f_{r+1}$.

Then clearly $\delta_r(f^*) = 0$.

Next, define a new set $f_{(\theta)}$ of functional values, such that $f_{j(\theta)} = \theta f_j + (1-\theta) f^*_j$ for any j from 1 to n. It is clear that all the $f_{j(\theta)}$'s must be linear in θ, since f_j and f^*_j are constants already fixed. From this it follows that $\mathfrak{m}_r(f_{(\theta)})$ is also linear in θ, as $\mathfrak{m}$ is linear in all the $f_{j(\theta)}$'s.

Since $\mathfrak{m}_r(f_{(\theta)})$ is linear in θ, there must exist a θ for which $\mathfrak{m}_r(f_{(\theta)}) = 0$. Let this value of θ be denoted by θ^{**}, and let f^{**} represent the corresponding set of functional values.

Then the statement that $\mathfrak{m}_r(f^{**}) = 0$ means that the best r^{th} order polynomial through $f^{**}_1\ f^{**}_2 \dots f^{**}_n$ has a leading coefficient of zero. I.e. it is of order (r–1) or lower. Thus it followes that $\delta_r(f^{**}) = \delta_{r-1}(f^{**})$.

Now the coefficients in $p_r(x)$ [and likewise in $p_{r-1}(x)$] can be seen to be linear in the functional values and hence in θ. Thus it follows that $\delta_r(f_{(\theta)})$ and $\delta_{r-1}(f_{(\theta)})$ are both quadratic in θ. The difference $\delta_{r-1}(f_{(\theta)}) - \delta_r(f_{(\theta)})$ is also quadratic in θ. But this difference can never be negative – (the best r^{th} order polynomial must always be at least as good a fit as the best $r-1^{th}$ order). Also $\delta_{r-1}(f_{(\theta)}) - \delta_r(f_{(\theta)})$ vanishes at $\theta = \theta^{**}$.

$$\therefore \quad \delta_{r-1}(f_{(\theta)}) - \delta_r(f_{(\theta)}) = K_1(\theta - \theta^{**})^2 \quad \text{where } K_1 \text{ is some constant.}$$

Again, $\mathfrak{m}_r(f_\theta)$ is as we have seen linear in θ, and by definition vanishes at $\theta = \theta^{**}$

$$\therefore \quad \mathfrak{m}_r(f_{(\theta)}) = K_2(\theta - \theta^{**}) \quad \text{where } K_2 \text{ is another constant.}$$

From this it follows at once that

$$\frac{\delta_{r-1}(f_{(\theta)}) - \delta_r(f_{(\theta)})}{\mathfrak{m}^2_r} = \frac{K_1}{K_2^2} = \text{a constant independent of } \theta.$$

As we could have defined f^* by any r+1 points (not necessarily the first ones), we can see that this constant must be the same whatever the values of the functional values, and in fact $(\delta_{r-1}(f) - \delta_r(f))/\mathfrak{m}^2_r(f)$ can depend only on the numbers n and r.

<u>Proof 4.</u> That $\dfrac{\delta_{r+1}(f) - \delta_r(f)}{\mathfrak{m}_r^2}$ is equal to $\binom{n+r}{2r+1}\Big/\binom{2r}{r}$

NOTE: In proof (3) above we have shown that $\dfrac{\delta_{r-1}(f) - \delta_r(f)}{\mathfrak{m}^2(r)}$ depends only on r and n, and not on the f's. Consequently we are at liberty to choose now what f's we wish

* * *

Let the f's be chosen so that $f_j = \dfrac{(-1)^{r-j+1}\binom{r}{j-1}}{r!}$ for $1 \leq j \leq r+1$,

and $f_j = 0$ for $r+1 < j < n$.

Then $\sum_{j=1}^{n} j^k f_j = 0$ for $k < r$ and $= 1$ for $k = r$.

From this it follows that $p_{r-1}(x) = 0$ and $\delta_{r-1}(f) = \sum_{j=1}^{n} (f_j)^2$.

For $p_r(x)$ we have the conditions

$$\left.\begin{array}{l} a_r S_{2r} + a_{r-1} S_{2r-1} + \dots + a_1 S_{r+1} + a_0 S_r = 1 \\ a_r S_{2r-1} + a_{r-1} S_{2r-2} + \dots + a_1 S_r + a_0 S_{r-1} = 0 \\ a_r S_{2r-2} + a_{r-1} S_{2r-3} + \dots + a_1 S_{r-1} + a_0 S_{r-2} = 0 \\ \hline a_r S_r + a_{r-1} S_{r-1} + \dots + a_1 S_1 + a_0 S_0 = 0 \end{array}\right\} \quad \dots (1)$$

But $$\delta_r = \sum_{j=1}^{n} (a_r j^r + a_{r-1} j^{r-1} + \dots + a_1 j + a_0 - f_j)^2$$

$$\begin{aligned} = {} & a_r(a_r S_{2r} + \dots + a_0 S_r) \\ & + a_{r-1}(a_r S_{2r-1} + \dots + a_0 S_{r-1}) \\ & \text{- - - - - - - - - - - -} \\ & + a_0(a_r S_r + \dots + a_0) \\ & - 2a_r \Sigma f_j j^r - 2a_{r-1} \Sigma f_j j^{r-1} - \dots - 2a_0 \Sigma f_j \\ & + \Sigma(f_j)^2 \end{aligned}$$

$$= a_r - 2a_r + \Sigma(f_j)^2$$

So that $\delta_{r-1} - \delta_r = a_r$.

But, from the conditions (1) above, it follows that

$$a_r = \begin{vmatrix} S_{2r-2} & \dots & S_{r-1} \\ \hline S_{r-1} & \dots & S_0 \end{vmatrix} \div \begin{vmatrix} S_{2r} & \dots & S_r \\ \hline S_r & \dots & S_0 \end{vmatrix}$$

$$= \frac{\Gamma(n+r-1)\ \Gamma(n-r-1)[\Gamma(r-1)]^4\ [\Gamma(n-1)]^2\ \Gamma(2r+1)}{[\Gamma(n-1)]^2\ \Gamma(2r-1)\ \Gamma(n+r)\Gamma(n-r-2)[\Gamma(r)]^4}$$

$$= \frac{(n-r-1)!\,(2r)!\ (2r+1)!}{(n+r)!\quad (r!)^4}$$

Again, the quantity Δ_1^r for the f's as defined, is

$\frac{1}{r!}\binom{2r}{r}$; Δ_2^r is $-\frac{1}{r!}\binom{2r}{r+1}$, and so on.

Hence $\mathfrak{m}_r$ is $$\frac{\left\{\binom{n-1}{r}\binom{r}{r}\binom{2r}{r}\right\} - \left\{\binom{n-2}{r}\binom{r+1}{r}\binom{2r}{r+1}\right\} \cdots\cdots \pm \left\{\binom{r}{r}\binom{n-1}{r}\binom{2r}{n-r-1}\right\}}{r!\ \binom{n+r}{2r+1}}$$

$$= \frac{2r!(2r+1)!(n-r-1)!}{(n+r)!\ (r!)^3} \sum_{j=1}^{\substack{j=r+1 \\ \text{or } j=n-r}} \frac{(-)^{j+1}\ (n-j)!}{(n-j-r)!(j-1)!(r-j+1)!}$$

and the summation is in all cases unity.

$$\therefore \quad \frac{\delta_{r-1} - \delta_r}{\mathfrak{m}_r^2} = \frac{(n-r-1)!(2r)!(2r+1)!}{(n+r)!\ (r!)^4} \div \left\{ \frac{2r!(2r+1)!(n-r-1)!}{(n+r)!\,(r!)^3} \right\}^2$$

$$= \frac{(n+r)!\ \ r!^2}{(2r+1)!(n-r-1)!\ 2r!} = \binom{n+r}{2r+1}\Big/\binom{2r}{r}$$

Q.E.D.

Proof 5. That $\sum_j \left\{ \sum_m (-1)^m \binom{r}{m}\binom{m+j-1}{r}\binom{n-j-m+r}{r}\right\}^2 = \binom{2r}{r}\binom{n+r}{2r+1}$.

Let $m+j = k$ and $n+r = s$

Then $\text{L.H.S.} = \sum_j \left\{ \sum_k (-1)^k \binom{r}{k-j}\binom{k-1}{r}\binom{s-k}{r}\right\}^2$

$$= \sum_j \sum_k \sum_l \left\{(-1)^{k-l}\binom{r}{k-j}\binom{r}{l-j}\binom{k-1}{r}\binom{l-1}{r}\binom{s-k}{r}\binom{s-l}{r}\right\}$$

We can sum at once over all j, for

$$\sum_j \binom{r}{k-j}\binom{r}{l-j} = \binom{2r}{r+k-l}$$

$$\therefore \quad \text{L.H.S.} = \sum_k \sum_l \left\{ (-1)^{k-1} \binom{2r}{r+k-l}\binom{k-1}{r}\binom{l-1}{r}\binom{s-k}{r}\binom{s-l}{r} \right\}$$

$$\text{Now} \quad \binom{2r}{r+k-l}\binom{k-1}{r}\binom{l-1}{r} = \binom{2r}{r}\binom{k-1}{r+k-l}\binom{l-1}{r+l-k}$$

At this stage it is useful to put $k=l+d$

$$\text{Thus L.H.S.} = \binom{2r}{r} \sum_l \sum_d \left\{ (-1)^d \binom{l+d-1}{r+d}\binom{l-1}{r-d}\binom{s-l-d}{r}\binom{s-l}{r} \right\}$$

Carrying out the summation first over all d, we get

$$\binom{2r}{r} \sum_l \binom{s-l}{r} \sum_d \left\{ (-1)^d \binom{l-1}{r-d}\binom{l+d-1}{r+d}\binom{s-l-d}{r} \right\}$$

which may be written as

$$\sum_l \binom{s-l}{r}\binom{l-1}{r} \sum_d \left\{ (-1)^d \binom{l+d-1}{r}\binom{s-l-d}{r}\binom{2r}{r+d} \right\}$$

Now it may be shown that $\displaystyle\sum_d (-1)^d \binom{a+d}{r}\binom{b-d}{r}\binom{2r}{r+d} = \binom{2r}{r}$

for all $a > r$, $b > r$.

Putting $a = l-1$, $b=s-l$, these inequalities are seen to apply. Consequently we are left with

$$\binom{2r}{r} \sum_l \binom{s-l}{r}\binom{l-1}{r} \{ 1 \}$$

and this reduces at once to $\binom{2r}{r}\binom{s}{2r+1}$

which = R.H.S.

[NOTE. We have omitted the limits of summation, since the summation is required for all values of the parameters giving nonzero contribution, and may in fact be taken from $-\infty$ to $+\infty$ for each parameter].

TABLES I and IA.

Weighting Factors to be Applied to Δ_j^r in Calculating m_r, and Sums of Weighting Factors.

Functional Values given at 3 points

w.f.	m_0	m_1	m_2
	1	2	1
	1	2	
	1		
SUMS	3	4	1

Functional Values given at 4 points

w.f. for	m_0	m_1	m_2	m_3
	1	3	3	1
	1	4	3	
	1	3		
	1			
SUMS	4	10	6	1

Functional Values given at 5 points

w.f. for	m_0	m_1	m_2	m_3	m_4
	1	4	6	4	1
	1	6	9	4	
	1	6	6		
	1	4			
	1				
SUMS	5	20	21	8	1

Functional Values given at 6 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5
	1	5	10	10	5	1
	1	8	18	16	5	
	1	9	18	10		
	1	8	10			
	1	5				
	1					
SUMS	6	35	56	36	10	1

Functional Values given at 7 points

w. f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6
	1						
		6					
	1		15				
		10		20			
	1		30		15		
		12		40		6	
	1		36		25		1
		12		40		6	
	1		30		15		
		10		20			
	1		15				
		6					
	1						
SUMS	7	56	126	120	55	12	1

Functional Values given at 8 points

w. f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
	1							
		7						
	1		21					
		12		35				
	1		45		35			
		15		80		21		
	1		60		75		7	
		16		100		36		1
	1		60		75		7	
		15		80		21		
	1		45		35			
		12		35				
	1		21					
		7						
	1							
SUMS	8	84	252	330	220	78	14	1

Functional Values given at 9 points.

w. f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
	1								
		8							
	1		28						
		14		56					
	1		63		70				
		18		140		56			
	1		90		175		28		
		20		200		126		8	
	1		100		225		49		1
		20		200		126		8	
	1		90		175		28		
		18		140		56			
	1		63		70				
		14		56					
	1		28						
		8							
	1								
SUMS	9	120	462	792	715	364	105	16	1

TABLES I & IA Continued

Functional Values given at 10 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
	1	9								
	1	16	36	84						
	1	21	84	224	126	126				
	1	24	126	350	350	336	84	36		
	1	25	150	400	525	441	196	64	9	1
	1	24	150	350	525	336	196	36	9	
	1	21	126	224	350	126	84			
	1	16	84	84	126					
	1	9	36							
	1									
SUMS	10	165	792	1716	2002	1365	560	136	18	1

Functional Values given at 11 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	10									
	1	18	45	120							
	1	24	108	336	210	252					
	1	28	168	560	630	756	210	120			
	1	30	210	700	1050	1176	588	288	45	10	
	1	30	225	700	1225	1176	784	288	81	10	1
	1	28	210	560	1050	756	588	120	45		
	1	24	168	336	630	252	210				
	1	18	108	120	210						
	1	10	45								
	1										
SUMS	11	220	1287	3432	5005	4368	2380	816	171	20	1

Functional Values given at 12 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	11									
	1	20	55	165							
	1	27	135	480	330	462					
	1	32	216	840	1050	1512	462	330			
	1	35	280	1120	1890	2646	1470	960	165	55	
	1	36	315	1225	2450	3136	2352	1296	405	100	11
	1	35	315	1120	2450	2646	2352	960	405	55	11
	1	32	280	840	1890	1512	1470	330	165		
	1	27	216	480	1050	462	462				
	1	20	135	165	330						
	1	11	55								
	1										
SUMS	12	286	2002	6435	11440	12376	8568	3876	1140	210	22

TABLES I & IA Continued

Functional Values given at 13 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	12									
	1	22	66	220							
	1	30	165	660	495	792					
	1	36	270	1200	1650	2772	924	792			
	1	40	360	1680	3150	5292	3234	2640	495	220	
	1	42	420	1960	4410	7056	5880	4320	1485	550	66
	1	42	441	1960	4900	7056	7056	4320	2025	550	121
	1	40	420	1680	4410	5292	5880	2640	1485	220	66
	1	36	360	1200	3150	2772	3234	792	495		
	1	30	270	660	1650	792	924				
	1	22	165	220	495						
	1	12	66								
	1										
SUMS	13	364	3003	11440	24310	31824	27132	15504	5985	1540	253

Functional Values given at 14 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	13									
	1	24	78	286							
	1	33	198	880	715	1287					
	1	40	330	1650	2475	4752	1716	1716			
	1	45	450	2400	4950	9702	6468	6336	1287	715	
	1	48	540	2940	7350	14112	12936	11880	4455	2200	286
	1	49	588	3136	8820	15876	17640	14400	7425	3025	726
	1	48	588	2940	8820	14112	17640	11880	7425	2200	726
	1	45	540	2400	7350	9702	12936	6336	4455	715	286
	1	40	450	1650	4950	4752	6468	1716	1287		
	1	33	330	880	2475	1287	1716				
	1	24	198	286	715						
	1	13	78								
	1										
SUMS	14	455	4368	19448	48620	75582	77520	54264	28334	8855	2024

TABLES I & IA Continued

Functional Values given at 15 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	14	91	364	1001	2002	3003	3432	3003	2002	1001
	1	26	234	1144	3575	7722	12012	13728	11583	7150	3146
	1	36	396	2200	7425	16632	25872	28512	22275	12100	4356
	1	44	550	3300	11550	25872	38808	39600	27225	12100	3146
	1	50	675	4200	14700	31752	44100	39600	22275	7150	1001
	1	54	756	4704	15876	31752	38808	28512	11583	2002	
	1	56	784	4704	14700	25872	25872	13728	3003		
	1	56	756	4200	11550	16632	12012	3432			
	1	54	675	3300	7425	7722	3003				
	1	50	550	2200	3575	2002					
	1	44	396	1144	1001						
	1	36	234	364							
	1	26	91								
	1	14									
	1										
SUMS	15	560	6188	31824	92378	167960	203490	170544	100947	42504	12650

Functional Values given at 16 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	15	105	455	1365	3003	5005	6435	6435	5005	3003
	1	28	273	1456	5005	12012	21021	27456	27027	20020	11011
	1	39	468	2860	10725	27027	48048	61776	57915	39325	18876
	1	48	660	4400	17325	44352	77616	95040	81675	48400	18876
	1	55	825	5775	23100	58212	97020	108900	81675	39325	11011
	1	60	945	6720	26460	63504	97020	95040	57915	20020	3003
	1	63	1008	7056	26460	58212	77616	61776	27027	5005	
	1	64	1008	6720	23100	44352	48048	27456	6435		
	1	63	945	5775	17325	27027	21021	6435			
	1	60	825	4400	10725	12012	5005				
	1	55	660	2860	5005	3003					
	1	48	468	1456	1365						
	1	39	273	455							
	1	28	105								
	1	15									
	1										
SUMS	16	680	8568	50388	167960	352716	497420	490314	346104	177100	65780

TABLES I & IA Continued

Functional Values given at 17 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	16	120	560	1820	4368	8008	11440	12870	11440	8008
	1	30	315	1820	6825	18018	35035	51480	57915	50050	33033
	1	42	546	3640	15015	42042	84084	123552	135135	110110	66066
	1	52	780	5720	25025	72072	144144	205920	212355	157300	81796
	1	60	990	7700	34650	99792	194040	261360	245025	157300	66066
	1	66	1155	9240	41580	116424	213444	261360	212355	110110	33033
	1	70	1260	10080	44100	116424	194040	205920	135135	50050	8008
	1	72	1296	10080	41580	99792	144144	123552	57915	11440	
	1	72	1260	9240	34650	72072	84084	51480	12870		
	1	70	1155	7700	25025	42042	35035	11440			
	1	66	990	5720	15015	18018	8008				
	1	60	780	3640	6825	4368					
	1	52	546	1820	1820						
	1	42	315	560							
	1	30	120								
	1	16									
	1										
SUMS	17	816	11628	77520	293930	705432	1144066	1307504	1081575	657800	296010

Functional Values given at 18 points

w.f. for	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
	1	17	136	680	2380	6188	12376	19448	24310	24310	19448
	1	32	360	2240	9100	26208	56056	91520	115830	114400	88088
	1	45	630	4550	20475	63063	140140	231660	289575	275275	198198
	1	56	910	7280	35035	112112	252252	411840	495495	440440	286286
	1	65	1170	10010	50050	162162	360360	566280	637065	511225	286286
	1	72	1386	12320	62370	199584	426888	627264	637065	440440	198198
	1	77	1540	13860	69300	213444	426888	566280	495495	275275	88088
	1	80	1620	14400	69300	199584	360360	411840	289575	114400	19448
	1	81	1620	13860	62370	162162	252252	231660	115830	24310	
	1	80	1540	12320	50050	112112	140140	91520	24310		
	1	77	1386	10010	35035	63063	56056	19448			
	1	72	1170	7280	20475	26208	12376				
	1	65	910	4550	9100	6188					
	1	56	630	2240	2380						
	1	45	360	680							
	1	32	136								
	1	17									
	1										
SUMS	18	969	15504	116280	497420	1352078	2496114	3268760	3124550	2220075	1184040

Functional Values given at 19 points

w. f. for m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
1	18	153	816	3060	8568	18564	31824	43758	48620	43758
1	34	408	2720	11900	37128	86632	155584	218790	243100	213928
1	48	720	5600	27300	91728	224224	411840	579150	629200	528528
1	60	1050	9100	47775	168168	420420	772200	1061775	1101100	858858
1	70	1365	12740	70070	252252	630630	1132560	1486485	1431430	1002001
1	78	1638	16016	90090	324324	792792	1359072	1656369	1431430	858858
1	84	1848	18480	103950	365904	853776	1359072	1486485	1101100	528528
1	88	1980	19800	108900	365904	792792	1132560	1061775	629200	213928
1	90	2025	19800	103950	324324	630630	772200	579150	243100	43758
1	90	1980	18480	90090	252252	420420	411840	218790	48620	
1	88	1848	16016	70070	168168	224224	155584	43758		
1	84	1638	12740	47775	91728	86632	31824			
1	78	1365	9100	27300	37128	18564				
1	70	1050	5600	11900	8568					
1	60	720	2720	3060						
1	48	408	816							
1	34	153								
1	18									
1										
SUMS 19	1140	20349	170544	817190	2496144	5200300	7726160	8436285	6906900	4292145

Functional Values given at 20 points

w. f. for m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
1	19	171	969	3876	11628	27132	50388	75582	92378	92378
1	36	459	3264	15300	51408	129948	254592	393822	486200	481338
1	51	816	6800	35700	129948	346528	700128	1093950	1337050	1283568
1	64	1200	11200	63700	244608	672672	1372800	2123550	2516800	2290288
1	75	1575	15925	95550	378378	1051050	2123550	3185325	3578575	3006003
1	84	1911	20384	126126	504504	1387386	2718144	3864861	4008004	3006003
1	91	2184	24024	150150	594594	1585584	2944656	3864861	3578575	2290288
1	96	2376	26400	163350	627264	1585584	2718144	3185325	2516800	1283568
1	99	2475	27225	163350	594594	1387386	2123550	2123550	1337050	481338
1	100	2475	26400	150150	504504	1051050	1372800	1093950	486200	92378
1	99	2376	24024	126126	378378	672672	700128	393822	92378	
1	96	2184	20384	95550	244608	346528	254592	75582		
1	91	1911	15925	63700	129948	129948	50388			
1	84	1575	11200	35700	51408	27132				
1	75	1200	6800	15300	11628					
1	64	816	3264	3876						
1	51	459	969							
1	36	171								
1	19									
1										
SUMS 20	1330	26334	245157	1307504	4457400	10400600	17383860	21474180	20030010	14307150

TABLE II

Summary Table of Sums of the Weighting Factors.

Number of points at which functional values are given. Sum of w.f. for $\mathfrak{m}_0$	Sum of w.f. for $\mathfrak{m}_1$	Sum of w.f. for $\mathfrak{m}_2$	Sum of w.f. for $\mathfrak{m}_3$	Sum of w.f. for $\mathfrak{m}_4$	Sum of w.f. for $\mathfrak{m}_5$	Sum of w.f. for $\mathfrak{m}_6$	Sum of w.f. for $\mathfrak{m}_7$	Sum of w.f. for $\mathfrak{m}_8$	Sum of w.f. for $\mathfrak{m}_9$	Sum of w.f. for $\mathfrak{m}_{10}$
1										
2	1									
3	4	1								
4	10	6	1							
5	20	21	8	1						
6	35	56	36	10	1					
7	56	126	120	55	12	1				
8	84	252	330	220	78	14	1			
9	120	462	792	715	364	105	16	1		
10	165	792	1716	2002	1365	560	136	18	1	
11	220	1287	3432	5005	4368	2380	816	171	20	1
12	286	2002	6435	11440	12376	8568	3876	1140	210	22
13	364	3003	11440	24310	31824	27132	15504	5985	1540	253
14	455	4368	19448	48620	75582	77520	54264	26334	8855	2024
15	560	6188	31824	92378	167960	203490	170544	100947	42504	12650
16	680	8568	50388	167960	352716	497420	490314	346104	177100	65780
17	816	11628	77520	293930	705432	1144066	1307504	1081575	657800	296010
18	969	15504	116280	497420	1352078	2496144	3268760	3124550	2220075	1184040
19	1140	20349	170544	817190	2496144	5200300	7726160	8436285	6906900	4292145
20	1330	26334	245157	1307504	4457400	10400600	17383860	21474180	20030010	14307150

General form: The sum of the weighting factors for $\mathfrak{m}_r$, when functional values are given at n points, is the binomial coefficient $\binom{n+r}{2r+1}$.

TABLE III

$\delta_{r-1} - \delta_r$ is given by $K\mathfrak{m}_r^2$, where the constant K is equal to the sum of the weighting factors for $\mathfrak{m}_r$, given in the table above, divided by the binomial coefficient $\binom{2r}{r}$, a function of r only and independent of n. This is tabulated below for r up to 10.

r	0	1	2	3	4	5	6	7	8	9	10
$\binom{2r}{r}$	1	2	6	20	70	252	924	3432	12870	49620	184756

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